Three problems on sequences defined recursively.
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1. Sequences.

1. Let $a_0, a_1, a_2, ..$ be the sequence defined by recurrence as

$$\begin{cases} a_0 = 2, a_1 = 3; \\ a_{n+1} = \dfrac{a_n + a_{n-1}}{6} \text{ for } n \geq 1. \end{cases}$$

(**i**) Show that for all $n \geq 2$ we have $a_n = b_n/6^{n-1}$ with $b_n \equiv -1 \pmod 6$.
(**ii**) For each $n \geq 0$, set $c_n = 5a_n + (-1)^n 4/3^{n-1}$. Show that for all $n \geq 0$ we have $c_n = 22 \cdot 2^{-n}$.

2. Let $a_0, a_1, a_2, ..$ be the sequence defined by recurrence as

$$\begin{cases} a_0 = 0, a_1 = 1; \\ a_{n+1} = 5a_n - 6a_{n-1} \text{ for } n \geq 1. \end{cases}.$$

Show that
(**i**) $(a_n, 6) = 1$ for all $n > 0$;
(**ii**) $5 \mid a_n$ if n is even.

3. Let $a_0, a_1, a_2, ..$ be the sequence defined by recurrence as

$$\begin{cases} a_1 = 1, a_2 = 2; \\ a_{n+1} = \dfrac{1}{2} a_n + a_{n-1} \text{ for } n \geq 2. \end{cases}.$$

(**i**) Show that $a_{n+1} \geq a_n$ for all $n \geq 1$;
(**ii**) Show that $a_{2n+2} = 9a_{2n}/4 - a_{2n-2}$ for all $n \geq 2$.

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1(i). Since characteristic equation $6x^2 - x - 1 = 0$ have roots $x_1 = 1/2, x_2 = -\frac{1}{3}$ then $a_n = c_1\left(\frac{1}{2}\right)^n + c_2\left(-\frac{1}{3}\right)^n$. Using initial conditions $a_0 = 2, a_1 = 3$ we obtain $c_1 = \frac{22}{5}, c_2 = -\frac{12}{2}$ and, therefore, $a_n = \frac{22}{5}\left(\frac{1}{2}\right)^n - \frac{12}{2}\left(-\frac{1}{3}\right)^n = \frac{1}{5 \cdot 6^{n-1}}\left(11 \cdot 3^{n-1} + (-1)^{n-1} \cdot 2^{n+1}\right)$.

Let $p_n := 11 \cdot 3^{n-1} + (-1)^{n-1} \cdot 2^{n+1}, n \in \mathbb{N}$. First we will prove that p_n is divisible by 5. Indeed, since $11 \equiv 1 \pmod 5, 3 \equiv -2 \pmod 5$, then $p_n \equiv \left((-2)^{n-1} + (-1)^{n-1} \cdot 2^{n+1}\right) \pmod 5 \equiv (-2)^{n-1}(1+4) \pmod 5 \equiv 0 \pmod 5$. Also, since $p_n \equiv 1 \pmod 2$ and $p_n \equiv 1 \pmod 3$ then $p_n \equiv 1 \pmod 6$. Thus, $b_n = \frac{p_n}{5} \in \mathbb{Z}$ and $b_n \equiv 1 \pmod 6$ (because $b_n - 1 \equiv 5(b_n - 1) \pmod 6$ and $5(b_n - 1) = p_n - 5 = (p_n - 1) + 6$ is divisible by 6).

1(ii). $c_n = 5a_n + (-1)^n 4/3^{n-1} = \frac{1}{6^{n-1}}\left(11 \cdot 3^{n-1} + (-1)^{n-1} \cdot 2^{n+1}\right) + \frac{(-1)^n 4}{3^{n-1}} = \left(\frac{11 \cdot 3^{n-1}}{6^{n-1}} + \frac{(-1)^{n-1} \cdot 2^{n+1}}{6^{n-1}}\right) + \frac{(-1)^n 4}{3^{n-1}} = \frac{11}{2^{n-1}} + \frac{(-1)^{n-1} \cdot 4}{3^{n-1}} + \frac{(-1)^n 4}{3^{n-1}} = \frac{22}{2^n}$.

2(i). $\gcd(a_1, 6) = \gcd(1, 6) = 1$ and $\gcd(a_{n+1}, 6) = \gcd(5a_n - 6a_{n-1}, 6) =$

$\gcd((5a_n - 6a_{n-1}) + 6a_{n-1}, 6) = \gcd(5a_n, 6) = \gcd(a_n, 6)$ since $\gcd(5,6) = 1$.
Hence, by Math Induction $\gcd(a_n, 6) = 1$ for any $n \geq 1$.
2(**ii**). Since $a_{n+1} - 5a_n + 6a_{n-1} = 0$ and $a_{n+2} - 5a_{n+1} + 6a_n = 0$ then
$a_{n+2} - 5a_{n+1} + 6a_n + 5(a_{n+1} - 5a_n + 6a_{n-1}) = 0 \Leftrightarrow a_{n+2} - 19a_n + 30a_{n-1} = 0$
and, therefore, $a_{n+2} \equiv 19a_n (\bmod 5), n \in \mathbb{N} \cup \{0\}$. Since $a_0 = 0$ and for any $n \in \mathbb{N} \cup \{0\}$
assuming $a_{2n} \equiv 0 (\bmod 5)$ we obtain $a_{2n+2} \equiv 19a_{2n} (\bmod 5) \equiv 0 (\bmod 5)$.
Thus, by Math Induction $a_{2n} \equiv 0 (\bmod 5)$ for any $n \in \mathbb{N} \cup \{0\}$.

3(**i**) Since $a_3 = \frac{1}{2} \cdot 2 + 1 = 2 = a_2 > a_1$ and for any $n \geq 2$ assuming
$a_{n+1} \geq a_n \geq a_{n-1}$ we obtain $a_{n+2} = \frac{1}{2}a_{n+1} + a_n \geq \frac{1}{2}a_n + a_{n-1} = a_{n+2}$
then by Math Induction $a_{n+1} \geq a_n$ for all $n \geq 1$.
3(**ii**) Since $a_{n+1} - \frac{1}{2}a_n - a_{n-1} = 0$, $a_{n+2} - \frac{1}{2}a_{n+1} - a_n = 0$ and $a_n - \frac{1}{2}a_{n-1} - a_{n-2} = 0$
then $a_{n+2} - \frac{1}{2}a_{n+1} - a_n + \frac{1}{2}\left(a_{n+1} - \frac{1}{2}a_n - a_{n-1}\right) = 0 \Leftrightarrow a_{n+2} - \frac{5}{4}a_n - \frac{1}{2}a_{n-1} = 0$
and $a_{n+2} - \frac{5}{4}a_n - \frac{1}{2}a_{n-1} - \left(a_n - \frac{1}{2}a_{n-1} - a_{n-2}\right) = 0 \Leftrightarrow a_{n+2} - \frac{9}{4}a_n + a_{n-2} = 0, n \geq 3$.
Hence, $a_{2n+2} - \frac{9}{4}a_{2n} + a_{2n-2} = 0, n \geq 2$.